TENTAMEN IMAGE PROCESSING

24-06-2008

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A FORMULA SHEET IS INCLUDED ON PAGES 3-4

Put your name on all pages which you hand in, and number them. Write the total number of pages you hand in on the first page. Write clearly and not with pencil or red pen. You can answer in English or Dutch. ***Always motivate your answers.*** *You get 10 points for free. Success!*

Problem 1 (25 pt)
Consider a binary image X and a structuring element B.

a. Show that the erosion of X by the structuring element B is a subset of X when the origin $(0,0)$ is contained in B. For the case that $(0,0)$ is not contained in B, give an example where $X \ominus B$ is disjoint with X.

b. Consider the image in Figure 1. Give an algorithm involving morphological operations to remove all foreground (i.e., white) image components except the square center region, which should be preserved entirely. Motivate why your algorithm gives the correct result.

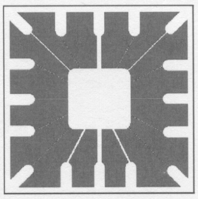

Figure 1: *Example image.*

Problem 2 (20 pt)
In this problem we consider histogram equalisation of a grey scale image with discrete grey levels.

a. What is the goal of histogram equalisation?

b. Let the image $f(x,y)$ have N pixels and grey level range $[0, L-1]$. Give a formula for the output $\mathcal{O}(f(x,y))$ of the histogram equalisation operation in terms of the cumulative histogram of the input image.

c. Explain why, in general, the discrete histogram equalisation technique does not yield the result one expects for continuous intensity levels.

(*continue on page 2*)

Problem 3 (25 pt)
A linear shift-invariant image degradation can be modelled by the equation

$$g(x,y) = (h * f)(x,y) + \eta(x,y) \tag{1}$$

where $g(x,y)$ is the degraded image, $f(x,y)$ is the ideal (unperturbed) image, $h(x,y)$ is the point spread function and $\eta(x,y)$ is the noise.

a. In the frequency domain equation (1) has the following form:

$$G(u,v) = H(u,v)F(u,v) + N(u,v)$$

Give a derivation of this formula.

To recover an estimate $\widehat{f}(x,y)$ of the ideal image from the perturbed image $g(x,y)$ one uses deconvolution.

b. The simplest form of deconvolution is inverse filtering. In the frequency domain, inverse filtering is expressed by the formula:

$$\widehat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

Explain why this filter is not useful in practice.

c. A better way of deconvolution is provided by the parametric Wiener filter or pseudo-inverse filter. Explain (in words or by formulas) the main idea behind this approach.

Problem 4 (20 pt)
Consider a binary image which contains an object with holes that have to be filled. This can be done with an iteration of the form:

$$\begin{aligned} X_0 &= F \\ X_k &= (X_{k-1} \oplus B) \cap G, \quad k = 1,2,3,\ldots \end{aligned} \tag{2}$$

until stability. Here F is called the marker image, and G is called the mask image.

a. Explain how the marker image F, the mask image G and the structuring element B have to be chosen.

b. Describe how the iterative process of equation (2) leads to hole filling.

c. What will be the output of the algorithm when the starting point is on the boundary of the object?

d. What will be the output of the algorithm when the starting point is outside the boundary of the object?

Formula sheet

Co-occurrence matrix $g(i,j) = \{\text{no. of pixel pairs with grey levels } (z_i, z_j) \text{ satisfying predicate } Q\}$, $1 \leq i,j \leq L$

Convolution, 2-D discrete $(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, h(x-m, y-n)$, for $x = 0,1,2,\ldots,M-1$, $y = 0,1,2,\ldots,N-1$

Convolution Theorem, 2-D discrete $\mathcal{F}\{f \star h\}(u,v) = F(u,v)\, H(u,v)$

Distance measures Euclidean: $D_e(p,q) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}$, City-block: $D_4(p,q) = |p_1 - q_1| + |p_2 - q_2|$, Chessboard: $D_8(p,q) = \max(|p_1-q_1|, |p_2-q_2|)$

Entropy, source $H = -\sum_{j=1}^{J} P(a_j) \log P(a_j)$

Entropy, estimated for L-level image: $\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k) \log_2 p_r(r_k)$

Error, root-mean square $e_{\mathrm{rms}} = \left[\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\widehat{f}(x,y) - f(x,y)\right)^2\right]^{\frac{1}{2}}$

Exponentials $e^{ix} = \cos x + i \sin x$; $\cos x = (e^{ix} + e^{-ix})/2$; $\sin x = (e^{ix} - e^{-ix})/2i$

Filter, inverse $\widehat{\mathbf{f}} = \mathbf{f} + \mathbf{H}^{-1}\mathbf{n}$, $\widehat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$

Filter, parametric Wiener $\widehat{\mathbf{f}} = (\mathbf{H}^t\mathbf{H} + K\,\mathbf{I})^{-1}\, \mathbf{H}^t\mathbf{g}$, $\widehat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + K}\right] G(u,v)$

Fourier series of signal with period T: $f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\frac{2\pi n}{T}t}$, with Fourier coefficients: $c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\frac{2\pi n}{T}t}\, dt, \quad n = 0, \pm 1, \pm 2, \ldots$

Fourier transform 1-D (continuous) $F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\mu t}\, dt$

Fourier transform 1-D, inverse (continuous) $f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{i2\pi\mu t}\, d\mu$

Fourier Transform, 2-D Discrete $F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-i2\pi(u\,x/M + v\,y/N)}$ for $u = 0,1,2,\ldots,M-1$, $v = 0,1,2,\ldots,N-1$

Fourier Transform, 2-D Inverse Discrete $f(x,y) = \frac{1}{M\,N} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{i2\pi(u\,x/M + v\,y/N)}$ for $x = 0,1,2,\ldots,M-1$, $y = 0,1,\ldots,N-1$

Fourier spectrum Fourier transform of $f(x,y)$: $F(u,v) = R(u,v) + i\,I(u,v)$, Fourier spectrum: $|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$, phase angle: $\phi(u,v) = \arctan\left(\frac{I(u,v)}{R(u,v)}\right)$

Gaussian function mean μ, variance σ^2: $G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$

Gradient $\nabla f(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

Histogram $h(m) = \#\{(x,y) \in D : f(x,y) = m\}$. Cumulative histogram: $\mathsf{P}(\ell) = \sum_{m=0}^{\ell} h(m)$

Impulse, discrete $\delta(0) = 1, \delta(x) = 0$ for $x \in \mathbb{N}\backslash\{0\}$

Impulse, continuous $\delta(\infty) = 1, \delta(x) = 0$ for $x \neq 0$, with $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$

Impulse train $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$, with Fourier transform $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$

Laplacian $\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Laplacian-of-Gaussian $\nabla^2 G_\sigma(x,y) = -\frac{2}{\pi\sigma^4}\left(1 - \frac{r^2}{2\sigma^2}\right) e^{-r^2/2\sigma^2} \quad (r^2 = x^2 + y^2)$

Morphology

Dilation $\delta_A(X) = X \oplus A = \bigcup_{a \in A} X_a = \bigcup_{x \in X} A_x = \{h \in E : \check{A}_h \cap X \neq \emptyset\}$,
where $X_h = \{x + h : x \in X\}$, $h \in E$ and $\check{A} = \{-a : a \in A\}$

Erosion $\varepsilon_A(X) = X \ominus A = \bigcap_{a \in A} X_{-a} = \{h \in E : A_h \subseteq X\}$

Opening $\gamma_A(X) = X \circ A := (X \ominus A) \oplus A = \delta_A \varepsilon_A(X)$

Closing $\phi_A(X) = X \bullet A := (X \oplus A) \ominus A = \varepsilon_A \delta_A(X)$

Hit-or-miss transform $X \circledast (B_1, B_2) = (X \ominus B_1) \cap (X^c \ominus B_2)$

Thinning $X \otimes B = X \backslash (X \circledast B)$, **Thickening** $X \odot B = X \cup (X \circledast B)$

Morphological reconstruction Marker F, mask G, structuring element B:
$X_0 = F,\ X_k = (X_{k-1} \oplus B) \cap G, \quad k = 1, 2, 3, \ldots$

Morphological skeleton Image X, structuring element B: $SK(X) = \bigcup_{n=0}^{N} S_n(X)$,
$S_n(X) = X \underset{n}{\ominus} B \setminus (X \underset{n}{\ominus} B) \circ B$, $S_0(X) = X$, with N the largest integer such that $S_N(X) \neq \emptyset$

Grey value dilation $(f \oplus b)(x, y) = \max\limits_{(s,t) \in B} [f(x - s, y - t) + b(s, t)]$

Grey value erosion $(f \ominus b)(x, y) = \min\limits_{(s,t) \in B} [f(x + s, y + t) - b(s, t)]$

Grey value opening $f \circ b = (f \ominus b) \oplus b$

Grey value closing $f \bullet b = (f \oplus b) \ominus b$

Morphological gradient $g = (f \oplus b) - (f \ominus b)$

Top-hat filter $T_{\text{hat}} = f - (f \circ b)$, **Bottom-hat filter** $B_{\text{hat}} = (f \bullet b) - f$

Sampling of continuous function $f(t)$: $\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$.
Fourier transform of sampled function: $\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$

Sampling theorem Signal $f(t)$, bandwidth $\mu_{\max}$: If $\frac{1}{\Delta T} \geq 2\mu_{\max}$, $f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\, \text{sinc}\left[\frac{t - n\Delta T}{n\Delta T}\right]$.

Sampling: downsampling by a factor of 2: $\downarrow_2 (a_0, a_1, a_2, \ldots, a_{2N-1}) = (a_0, a_2, a_4, \ldots, a_{2N-2})$

Sampling: upsampling by a factor of 2: $\uparrow_2 (a_0, a_1, a_2, \ldots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \ldots, a_{N-1}, 0)$

Set, circularity ratio $R_c = \frac{4\pi A}{P^2}$ of set with area A, perimeter P

Set, diameter $\text{Diam}(B) = \max\limits_{i,j} [D(p_i, p_j)]$ with p_i, p_j on the boundary B and D a distance measure

Sinc function $\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$ when $x \neq 0$, and $\text{sinc}\,(0) = 1$

Spatial moments of an $M \times N$ image $f(x, y)$: $m_{pq} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} x^p\, y^q\, f(x, y), \quad p, q = 0, 1, 2, \ldots$

Statistical moments of distribution $p(i)$: $\mu_n = \sum_{i=0}^{L-1} (i - m)^n\, p(i)$, $m = \sum_{i=0}^{L-1} i\, p(i)$

Signal-to-noise ratio, mean-square $\text{SNR}_{\text{rms}} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \hat{f}(x,y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\hat{f}(x,y) - f(x,y)\right)^2}$

Wavelet decomposition with low pass filter h_ϕ, band pass filter h_ψ. For $j = 1, \ldots, J$:
Approximation: $c_j = \mathbf{H} c_{j-1} = \downarrow_2 (h_\phi * c_{j-1})$; Detail: $d_j = \mathbf{G} c_{j-1} = \downarrow_2 (h_\psi * c_{j-1})$

Wavelet reconstruction with low pass filter $\tilde{h}_\phi$, band pass filter $\tilde{h}_\psi$. For $j = J, J-1, \ldots, 1$:
$c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)$

Wavelet, Haar basis $h_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $h_\psi = \frac{1}{\sqrt{2}}(1, -1)$, $\tilde{h}_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $\tilde{h}_\psi = \frac{1}{\sqrt{2}}(1, -1)$